LARGE SAMPLE VARIANCES OF MAXIMUM LIKELIHOOD ESTIMATORS OF VARIANCE COMPONENTS IN THE 3-WAY NESTED CLASSIFICATION, RANDOM MODEL, WITH UNBALANCED DATA

J. W. Rudan and S. R. Searle
Biometrics Unit, Cornell University
Ithaca, N. Y., 14850

Abstract

Explicit expressions are presented for the elements of the information matrix of the variance components in a 3-way nested classification, random model, with normality and unbalanced data.

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Summary

Explicit expressions are presented for the elements of the information matrix of the variance components in a 3-way nested classification, random model, with normality and unbalanced data.

1. Introduction and Model

Searle [1970] developed a general method for obtaining, under normality conditions, the elements of the information matrix of the variance components of mixed models, with unbalanced data; in particular he displayed the results for the 2-way nested classification. This paper presents analogous results for the 3-way nested classification, random model, for the general case of unbalanced data.

The linear model for an observation is taken to be

$$y_{ijkl} = \mu + \alpha_i + \beta_{ij} + \gamma_{ijk} + e_{ijkl} \tag{1}$$

where y_{ijkl} is the l-th response within the k-th level of the γ-factor within the j-th level of the β-factor within the i-th level of the α-factor. This nesting is indicated in the observation identifiers which are taken to be

$$i = 1,2,\ldots,a;\; j = 1,2,\ldots,b_i;\; k = 1,2,\ldots,c_{ij};$$

and $l = 1,2,\ldots,n_{ijk}$.

The α's, β's, γ's and e's are assumed to be normally and independently distributed with zero means and variances σ^2_α, σ^2_β, σ^2_γ and σ^2_e respectively.

There is a total of $n_{\cdots} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \sum_{k=1}^{c_{ij}} n_{ijk}$ observations. Suppose they are written as a vector $\underset{\sim}{y}$ in lexicon order, i.e., ordered by l within k within j within i. Then, similar to the 2-way nested classification of Searle [1970], $\underset{\sim}{V}$, the variance-covariance matrix of $\underset{\sim}{y}$ can be written as

$$\underset{\sim}{V} = \sum_{i=1}^{a}{}^{+} \underset{\sim}{V}_i , \tag{2}$$

where $\sum^{+}$ denotes the operation of a direct sum of matrices. Each $\underset{\sim}{V}_i$ in (2) can be written in partitioned matrix form as

$$\underset{\sim}{V}_i = \left\{\underset{\sim}{V}_{ij,ij'}\right\} \quad \text{for} \quad j,j' = 1,2,\ldots,b_i , \tag{3}$$

where, for $j = j'$, the partitioned form of $\underset{\sim}{V}_{ij,ij}$ is

$$\underset{\sim}{V}_{ij,ij} = \left\{\delta_{kk'}\left(\sigma^2_e \underset{\sim}{I} + \sigma^2_\gamma \underset{\sim}{J}_{n_{ijk} \times n_{ijk}}\right) + \left(\sigma^2_\alpha + \sigma^2_\beta\right)\underset{\sim}{J}_{n_{ijk} \times n_{ijk'}}\right\} \text{ for } k,k' = 1,2,\ldots,c_{ij} \tag{4}$$

and for $j \neq j'$, the partitioned form of $\underset{\sim}{V}_{ij,ij'}$ is

$$\underset{\sim}{V}_{ij,ij'} = \left\{\sigma^2_\alpha \underset{\sim}{J}_{n_{ijk} \times n_{ij'k'}}\right\} \text{ for } k = 1,2,\ldots,c_{ij} \text{ ; } k' = 1,2,\ldots,c_{ij'} \, . \tag{5}$$

In (4) and (5) $\delta_{kk'}$ is the Kronecker delta and $\underset{\sim}{J}_{n_{ijk} \times n_{ij'k'}}$ is a matrix of order $n_{ijk} \times n_{ij'k'}$ having every element unity.

2. Method

When $\underset{\sim}{\sigma}^2$ is the vector of variance components in a linear model, i.e., $\underset{\sim}{\sigma}^2 = \left(\sigma^2_1, \sigma^2_2, \ldots, \sigma^2_f\right)$, and $\underset{\sim}{\tilde{\sigma}}^2$ the corresponding vector of maximum likelihood estimators, Searle [1970] has shown that the large sample variance-covariance matrix of $\tilde{\underline{\sigma}}^2$, namely var($\tilde{\underline{\sigma}}^2$), can be written as

$$\mathrm{var}(\tilde{\underline{\sigma}}^2) = 2\underline{T}^{-1} = 2\left\{t_{\sigma_r^2\sigma_s^2}\right\}^{-1} \text{ for } r,s = 1,2,\ldots,f. \tag{6}$$

A typical element of $\underline{T}$ is

$$t_{\sigma_r^2\sigma_s^2} = \mathrm{trace}\left(\underline{V}^{-1}\frac{\partial\underline{V}}{\partial\sigma_r^2}\ \underline{V}^{-1}\frac{\partial\underline{V}}{\partial\sigma_s^2}\right) \tag{7}$$

where $\dfrac{\partial\underline{V}}{\partial\sigma_r^2}$ is the matrix of partial derivatives of the elements of $\underline{V}$ with respect to σ_r^2 .

After deriving (7), Searle [1970] obtains explicit expressions for the elements of $\underline{T}$ for the 2-way nested classification with unbalanced data. Direct extension of the methods used there yields the analogous results for the 3-way nested classification, given in section 3 that follows. The algebraic manipulations involved in deriving these results are detailed in a lengthy appendix to this paper which is available on request from the authors. Development of the results relies heavily on lemma 2 of Urquhart [1962] (quoted in Searle [1970]), and on properties of direct sum matrices, for deriving $\underline{V}^{-1}$ of (7) from $\underline{V}$ defined by equations (2)-(5). Repetitive use is also made of the fact that $\underline{J}_{n\times n} = \underline{1}_n\underline{1}_n'$ where $\underline{1}_n$ is an n-vector of 1's, and that

$$(\underline{A} + \lambda\underline{u}\underline{v}')^{-1} = \underline{A}^{-1} - \frac{\lambda\underline{A}^{-1}\underline{u}\underline{v}'\underline{A}^{-1}}{1 + \lambda\underline{u}'\underline{A}^{-1}\underline{v}}$$

for which existence conditions hold for the matrices defined by (2) through (5).

3. Results

Presentation of the results for the 3-way nested classification is simplified by using the following notation, which is an extension of that used by Searle [1970].

$$\underset{\sim}{\sigma}^2 = \left(\sigma^2_\alpha,\ \sigma^2_\beta,\ \sigma^2_\gamma,\ \sigma^2_e\right) \equiv (\alpha,\ \beta,\ \gamma,\ e), \tag{8}$$

$$m_{ijk} = n_{ijk}\sigma^2_\gamma + \sigma^2_e\ , \tag{9}$$

$$A_{ijpq} = \sum_{k=1}^{c_{ij}} \frac{(n_{ijk})^p}{(m_{ijk})^q}\ , \tag{10}$$

$$p_{ij} = 1 + \sigma^2_\beta\ A_{ij11}\ , \tag{11}$$

and

$$q_i = 1 + \sigma^2_\alpha \sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij}\right)\ . \tag{12}$$

Then $\underset{\sim}{T}$ of (6) is the 4×4 symmetric matrix

$$\underset{\sim}{T} = \begin{bmatrix} t_{\alpha\alpha} & t_{\alpha\beta} & t_{\alpha\gamma} & t_{\alpha e} \\ t_{\alpha\beta} & t_{\beta\beta} & t_{\beta\gamma} & t_{\beta e} \\ t_{\alpha\gamma} & t_{\beta\gamma} & t_{\gamma\gamma} & t_{\gamma e} \\ t_{\alpha e} & t_{\beta e} & t_{\gamma e} & t_{ee} \end{bmatrix}\ ,$$

and its 10 different elements are as follows:

$$t_{\alpha\alpha} = \sum_{i=1}^{a} \left[\sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij} \right) \right]^2 / q_i^2 \quad , \tag{13}$$

$$t_{\alpha\beta} = \sum_{i=1}^{a} \left[\sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij} \right)^2 \right] / q_i^2 \quad , \tag{14}$$

$$t_{\alpha\gamma} = \sum_{i=1}^{a} \left[\sum_{j=1}^{b_i} A_{ij22}/p_{ij}^2 \right] / q_i^2 \quad , \tag{15}$$

$$t_{\alpha e} = \sum_{i=1}^{a} \left[\sum_{j=1}^{b_i} A_{ij12}/p_{ij}^2 \right] / q_i^2 \quad , \tag{16}$$

$$\begin{aligned} t_{\beta\beta} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \Big\{ & \left(A_{ij11}/p_{ij} \right)^2 - 2\sigma_\alpha^2 A_{ij11}^3/q_i p_{ij}^3 \\ & + \sigma_\alpha^4 \left(A_{ij11}/p_{ij} \right)^2 \left[\sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij} \right)^2 \right] / q_i^2 \Big\} \quad , \end{aligned} \tag{17}$$

$$\begin{aligned} t_{\beta\gamma} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \Big\{ & A_{ij22}/p_{ij}^2 - 2\sigma_\alpha^2 A_{ij11} A_{ij22}/q_i p_{ij}^3 \\ & + \sigma_\alpha^4 \left(A_{ij22}/p_{ij}^2 \right) \left[\sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij} \right)^2 \right] / q_i^2 \Big\} \quad , \end{aligned} \tag{18}$$

$$\begin{aligned} t_{\beta e} = \sum_{i=1}^{a} \sum_{j=1}^{b_i} \Big\{ & A_{ij12}/p_{ij}^2 - 2\sigma_\alpha^2 A_{ij11} A_{ij12}/q_i p_{ij}^3 \\ & + \sigma_\alpha^4 \left(A_{ij12}/p_{ij}^2 \right) \left[\sum_{j=1}^{b_i} \left(A_{ij11}/p_{ij} \right)^2 \right] / q_i^2 \Big\} \quad , \end{aligned} \tag{19}$$

$$t_{\gamma\gamma} = \sum_{i=1}^{a}\sum_{j=1}^{b_i}\Big\{A_{ij22} - 2\sigma_\alpha^2 A_{ij33}/q_i p_{ij}^2 - 2\sigma_\beta^2 A_{ij33}/p_{ij}$$

$$+ 2\sigma_\alpha^2\sigma_\beta^2 A_{ij22}^2/q_i p_{ij}^3 + \sigma_\beta^4\left(A_{ij22}/p_{ij}\right)^2$$

$$+ \sigma_\alpha^4\left(A_{ij22}/p_{ij}^2\right)\Big[\sum_{j=1}^{b_i} A_{ij22}/p_{ij}^2\Big]/q_i^2\Big\}\ , \qquad (20)$$

$$t_{\gamma e} = \sum_{i=1}^{a}\sum_{j=1}^{b_i}\Big\{A_{ij12} - 2\sigma_\alpha^2 A_{ij23}/q_i p_{ij}^2 - 2\sigma_\beta^2 A_{ij23}/p_{ij}$$

$$+ 2\sigma_\alpha^2\sigma_\beta^2 A_{ij12}A_{ij22}/q_i p_{ij}^3 + \sigma_\beta^4 A_{ij12}A_{ij22}/p_{ij}^2$$

$$+ \sigma_\alpha^4\left(A_{ij12}/p_{ij}^2\right)\Big[\sum_{j=1}^{b_i} A_{ij22}/p_{ij}^2\Big]/q_i^2\Big\}\ , \qquad (21)$$

and

$$t_{ee} = \sum_{i=1}^{a}\sum_{j=1}^{b_i}\Big\{A_{ij02} - 2\sigma_\alpha^2 A_{ij13}/q_i p_{ij}^2 - 2\sigma_\beta^2 A_{ij13}/p_{ij}$$

$$+ 2\sigma_\alpha^2\sigma_\beta^2 A_{ij12}^2/q_i p_{ij}^3 + \sigma_\beta^4\left(A_{ij12}/p_{ij}\right)^2$$

$$+ \sigma_\alpha^4\left(A_{ij12}/p_{ij}^2\right)\Big[\sum_{j=1}^{b_i} A_{ij12}/p_{ij}^2\Big]/q_i^2\Big\}$$

$$+ (n_{\cdots} - c_{\cdot\cdot})/\sigma_e^4\ . \qquad (22)$$

4. Validation

The above results have been partially validated in 2 major ways; by ensuring that they reduce both to those for the 2-way nested classification and to those for the balanced data case. The first way is to set either $\sigma_\alpha^2 = 0$, or $\sigma_\beta^2 = 0$, or $\sigma_\gamma^2 = 0$ and appropriately adjust the model and the factor subscripts so that the model reduces to the 2-way nested classification. With obvious adjustments to equations (8) through (12) in all 3 cases the results (11) through (20) then reduce to those given by Searle [1970]. For example, if $\sigma_\alpha^2 = 0$ then $t_{\beta\beta}$ given in (17) corresponds to $t_{\alpha\alpha}$ of (27) given in Searle [1970].

The second validation was to consider the balanced data wherein $n_{ijk} = n$ for all i, j, k, $c_{ij} = c$ for all i and j, and $b_i = b$ for all i . Results (13) through (21) then lead to the customary results for balanced data as indicated, for example, in Mahamunulu [1963].

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